CAPACITY OF GAUSSIAN CHANNEL

$$\max_{f_Y(y):\, E\{Y^2\} \le \sigma^2 + P} h(Y) = \frac{1}{2}\log_2 2\pi\sigma^2 e$$

Let us Focus on maximizing DE of output Y.

Consider:

$$\int_{-\infty}^{\infty} g_Y(y) \log_2\left(\frac{f_Y(y)}{g_Y(y)}\right) dy$$

x_i

PDF P_i

Continuous sum

$f_Y(y), g_Y(y)$

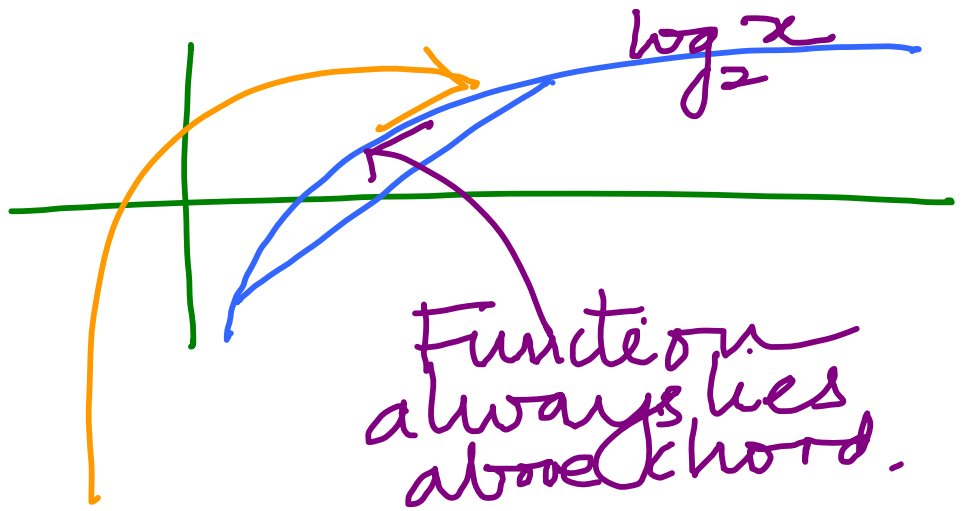
Two different Probability Density Functions for Y.

Jensen's Inequality

$$\sum_{i=0}^{M-1} P_i \log_2 x_i \leq \log_2\left(\sum_{i=0}^{M-1} P_i x_i\right)$$

Extend to integral:

$$\int_{-\infty}^{\infty} P_X(x) \log_2 g(x) \leq \log_2 \int_{-\infty}^{\infty} P_X(x) g(x) dx$$



Log is concave.

$$\int_{-\infty}^{\infty} g_Y(y) \log_2\left(\frac{F_Y(y)}{g_Y(y)}\right) dy$$

$$\leq \log_2\left(\int_{-\infty}^{\infty} g_Y(y) \cdot \frac{F_Y(y)}{g_Y(y)} dy\right)$$

$$= \log_2 \left(\int_{-\infty}^{\infty} f_Y(y)\,dy \right)$$

$$= \log_2 1 = 0 \cdot \quad f_Y(y) = \text{PDF}$$

$$\int_{-\infty}^{\infty} g_Y(y) \log_2 \left(\frac{f_Y(y)}{g_Y(y)} \right) dy \leq 0$$

$$\Rightarrow \int_{-\infty}^{\infty} -g_Y(y) \log_2 \left(\frac{g_Y(y)}{f_Y(y)} \right) dy$$

$$\leq 0 .$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} g_Y(y) \log_2\left(\frac{g_Y(y)}{f_Y(y)}\right) dy \geq 0}$$

Kullback – Leibler (KL) Divergence $D(g_Y \| f_Y)$

$g_Y(y)$ = Any arbitrary PDF of random variable Y with $E\{Y^2\} = \sigma^2$

$$f_Y(y) = \frac{1}{\sqrt{2\pi\tilde{\sigma}^2}} e^{-y^2/2\tilde{\sigma}^2}$$

Gaussian PDF
mean $= 0$
var $= \tilde{\sigma}^2$

KL Divergence:

$$0 \leq \int_{-\infty}^{\infty} g_Y(y) \log_2 \left(\frac{g_Y(y)}{f_Y(y)} \right) dy$$

$$= \int_{-\infty}^{\infty} g_Y(y) \log_2 g_Y(y)\, dy$$

$$= -h(g_Y)$$

DE of Y corresponding to PDF $g_Y(y)$.

$$- \int_{-\infty}^{\infty} g_Y(y) \log_2 \left(\frac{1}{F_Y(y)} \right) dy$$

$$F_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-y^2/2\sigma^2}$$

Focus now on this.

$$\frac{1}{F_Y(y)} = \sqrt{2\pi\tilde{\sigma}^2}\, e^{y^2/2\tilde{\sigma}^2}$$

$$\log_2 \frac{1}{F_Y(y)} = \frac{1}{2}\log_2 2\pi\tilde{\sigma}^2 + (\log_2 e)\frac{y^2}{2\tilde{\sigma}^2}$$

$$\int_{-\infty}^{\infty} g_Y(y) \log_2 \frac{1}{F_Y(y)}\, dy$$

$$= \int_{-\infty}^{\infty} g_Y(y) \left\{ \frac{1}{2}\log_2 2\pi\tilde{\sigma}^2 + (\log_2 e)\frac{y^2}{2\tilde{\sigma}^2} \right\} dy$$

$$= \left(\int_{-\infty}^{\infty} g_Y(y)\,dy \right) \frac{1}{2} \log_2 (2\pi\tilde{\sigma}^2) + \frac{\log_2 e}{2\tilde{\sigma}^2} \int_{-\infty}^{\infty} g_Y(y)\, y^2\, dy$$

$$= \frac{1}{2}\log_2 2\pi\tilde{\sigma}^2 + \log_2 e \cdot \frac{1}{2\tilde{\sigma}^2} \cdot \tilde{\sigma}^2$$

$$= \frac{1}{2}\log 2\pi\tilde{\sigma}^2 + \frac{1}{2}\log_2 e$$

$$= \frac{1}{2}\log_2 2\pi\tilde{\sigma}^2 e$$

DE of Gaussian source.

$$\int_{-\infty}^{\infty} g_Y(y) \log_2 \frac{1}{F_Y(y)} dy$$

$$= \frac{1}{2} \log_2 (2\pi\tilde{\sigma}^2 e)$$

DE of Gaussian source with var $= \tilde{\sigma}^2$

$$0 \leq \int_{-\infty}^{\infty} g_Y(y) \log_2 \left(\frac{g_Y(y)}{F_Y(y)} \right) dy$$

$$= -h(g_Y) + \frac{1}{2} \log_2 (2\pi\tilde{\sigma}^2 e)$$

$$\Rightarrow \boxed{\frac{1}{2}\log_2\left(2\pi\tilde{\sigma}^2 e\right) \geq h(g_Y).}$$

For a given power $= \tilde{\sigma}^2$ Gaussian source has maximum Differential Entropy

For any arbitrary PDF $g_Y(y)$: $\mathbb{E}[Y^2] = P$.

$$C = \max_{f_Y(y):\, E\{Y^2\} \le \sigma^2} h(Y) - \frac{1}{2}\log_2(2\pi\sigma^2 e)$$

$\max$ = Gaussian source

$$\max h(Y) = \frac{1}{2}\log_2 2\pi\tilde{\sigma}^2 e$$

$$= \frac{1}{2}\log_2 2\pi\tilde{\sigma}^2 e - \frac{1}{2}\log_2 2\pi\sigma^2 e$$

$$= \frac{1}{2}\log_2\left(\frac{\tilde{\sigma}^2}{\sigma^2}\right)$$

$$C = \frac{1}{2}\log_2\left(\frac{P+\sigma^2}{\sigma^2}\right)$$

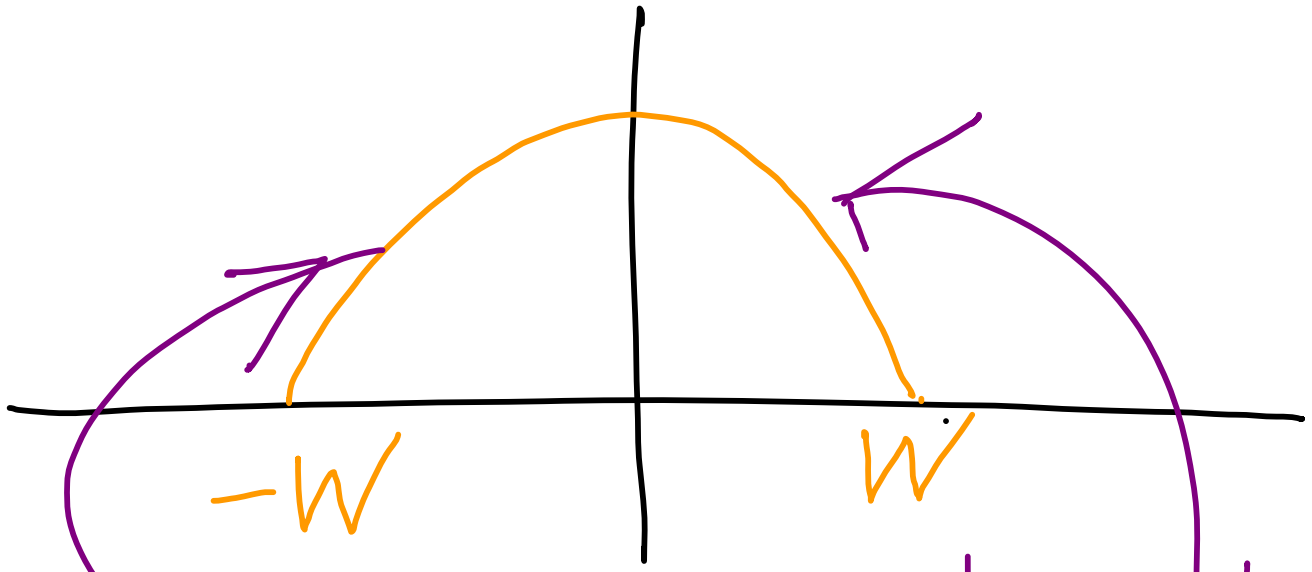
$$= \frac{1}{2}\log_2\left(1+\frac{P}{\sigma^2}\right)$$

$$\boxed{C = \frac{1}{2}\log_2(1+SNR)}$$

Fundamental result characterizing rate of information

very popular result for capacity of channel with Gaussian Noise.

transfer in bits/channel use.



channel of BW = 2W

Band limited to W

$\Rightarrow$ Nyquist Sampling rate

signal is completely characterized by samples at 2W.

symbols or samples/sec.

$\Rightarrow$ Maximum information rate

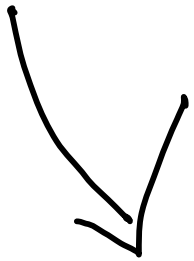
$$= \frac{1}{2}\log_2\left(1+\frac{P}{\sigma^2}\right) \text{ bits/symbol}$$

$$\times 2W \text{ symbols/sec}$$

$$\tilde{C} = W\log_2\left(1+\frac{P}{\sigma^2}\right)$$

Capacity in bits/sec.

Shannon–Hartley Law.



Fundamental Result which characterizes capacity of Gaussian channel in bits/sec.